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Wedge Flows of Power-law Fluids

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WEDGE FLOWS OF POWER-LAW FLUIDS

BY

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This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

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NOMENCLATURE

$a(x)$	-scale factor for y coordinate
$b(x)$	-scale factor for stream function
C_d	-coefficient of skin friction
$f(\eta)$	-dimensionless stream function
M	-material constant
n	-material constant
Re	-Reynolds number based on x
$U(x)$	-external flow velocity
u	-velocity component in the x direction
v	-velocity compenent in the y direction
x	-coordinate tangent to the body
y	-coordinate perpendicular to the body
ρ	-mass density of the fluid
τ_{xy}	-shear stress
ψ	-stream function
η	-stretched coordinate defined in Equation (2.12)

CHAPTER I

INTRODUCTION

Many solutions exist for the laminar boundary-layer flow of Newtonian fluids. This is verified in literature by Schlichting [11]. However, some fluids do not obey the Newtonian postulate that the stress tensor is directly proportional to the rate of deformation tensor. It is as likely, in fact, that non-Newtonian fluids could be pumped into an industrial complex as the more common Newtonian fluids. This is partly due to the growing use of synthetics and petroleum products in modern day living. The use of non-Newtonian fluids, however, does not have to be limited to problems of internal flow. The fluid in the boundary-layer of a ship's hull could be changed to a non-Newtonian fluid by injection of a water soluble chemical. As a result, the drag force experienced by the ship could be decreased. It is evident then, that the study of non-Newtonian fluids could yield valuable information.

Non-Newtonian fluids can be divided into three classes:

(1) Time-independent fluids for which the rate of deformation is dependent upon only the instantaneous shear stress. These models are sometimes called "purely viscous fluids."

(2) Time-dependent fluids for which the rate of deformation is a function of both the magnitude and duration of the shear stress.

(3) Viscoelastic materials in which the fluid has partial elastic recovery upon removal of a deforming shear stress.

Since the constitutive equation for time-dependent and viscoelastic fluids can become involved, purely viscous fluid models are more often easier to use. Purely viscous models, however, are able to predict the behavior of many non-Newtonian fluid flows.

A model of purely viscous fluid, used because of its simplicity and because it can be applied to a variety of fluid flows, is the power-law fluid model. The constitutive equation for this model is that the stress tensor is directly proportional to the nth power of the rate of deformation tensor. This model was used by Schowalter [12] to develop the two-dimensional boundary-layer theory of pseudo-plastic fluids. He found that the shear stress is proportional to the nth power of the velocity gradient and that similar solutions exist for Falkner-Skan type flows.

Acrivos et al [1] were the first to attempt a discussion of flow past external surfaces. They studied the momentum and heat transfer in laminar boundary-layer flows of power-law fluids and also solved the flat plate problem by both an integral method and a numerical method. Their results, however, indicated that the Pohlhausen integral method [11] is generally less accurate for studying power-law fluids than for Newtonian fluids. They also predicted that his inaccuracy might be increased when the surface became more complicated than a flat plate. In 1965, Berkovskii [3] obtained exact numerical solutions for the boundary-layer flow of pseudo-plastic fluids. His study dealt with the flow past a permeable flat plate and the flow near the stagnation point. Later, Lee and Ames [8], discussed similarity solutions for non-Newtonian fluid flows. They also applied a numerical technique to find the solution for forced convection of power-law fluids about a right angle wedge.

In this analysis a solution is obtained for steady, two-dimensional, incompressible, laminar, boundary-layer flows of power-law fluids. A similarity transformation is first used to convert the governing partial differential equation into an ordinary differential equation of Falkner-Skan type flows. A discussion of the external velocity functions which will permit similar flows is also presented here. The solution of the ordinary differential equation is found by a method of series expansion followed by use of the method of steepest descent (see appendix A). The solution obtained is a series of gamma functions which is usually divergent and Euler's transformation (see appendix B) is used to find the correct sum of the series.

This method was first used by Meksyn [9] to analyze boundary-layer flow of Newtonian fluids. He found the method to be powerful and very accurate. Hsu [7] also used this method to solve the problem of boundary-layer flow of power-law fluids past a semi-infinite flat plate. The results obtained were in excellent agreement with results found by numerical techniques [1 and 3].

CHAPTER II

GOVERNING EQUATIONS

The flow about a submerged body can be separated into two regions: a thin layer close to the body where frictional forces are of the same order of magnitude as the inertial forces, and a region outside this boundary-layer where the fluid can be considered ideal (i.e. friction can be neglected). Schlichting [11] developed the boundary-layer equations for Newtonian fluids by estimating the magnitude of each dimensionless term in the Navier-Stokes momentum equations.

Schowalter [12] used this same technique to find the boundary-layer equations for power-law fluids. He found that the governing boundary-layer equations of motion and continuity for steady, two-dimensional, incompressible, laminar flow past a submerged body are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{\partial \tau_{xy}}{\partial y} \qquad (2.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.2)$$

respectively, where ρ is the constant density, x and y are Cartesian coordinates measured tangent and normal to the body respectively, u and v are the components of the velocity vector in the x and y directions respectively, U is the external flow velocity, and τ_{xy} is the shear stress.

The constitutive equation for the power-law fluid model is

$$\tau_{xy} = M\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^n \tag{2.3}$$

where M and n are positive constants. In the boundary-layer however,

$$\frac{\partial u}{\partial y} >> \frac{\partial v}{\partial x}$$

and consequently equation (2.3) can be reduced to [12]

$$\tau_{xy} = M\left(\frac{\partial u}{\partial y}\right)^n \tag{2.4}$$

The boundary conditions for the boundary-layer equations are:

$$\text{at } y=0: \quad u=0, \quad v=0 \tag{2.5}$$

$$\text{and as } y \to \infty: \quad u \to U(x) \tag{2.6}$$

A stream function $\Psi(x,y)$ can be introduced from the continuity equation such that

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x} \tag{2.7}$$

By substituting equations (2.7) and (2.4) into equation (2.1) the result can be written as

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = U\frac{dU}{dx} + \frac{Mn}{\rho}\left[\Psi_{yy}\right]^{n-1}\Psi_{yyy} \tag{2.8}$$

where the subscripts denote partial differentiation. The boundary conditions (2.5) and (2.6) can also be found in terms of the stream function as

$$\Psi_y(x,0) = 0, \qquad \Psi_x(x,0) = 0 \tag{2.9}$$

$$\Psi_y(x,y\rightarrow\infty) = U(x) \tag{2.10}$$

A solution of the boundary-layer equations is not possible without the use of a similarity transformation. Schlichting [11] defined these 'similar' solutions as those for which the component u of the velocity vector has a property such that for any two different velocity profiles of $u(x,y)$, located at different coordinates x, the velocity profile of $u(x,y)$ would differ only by a scale factor in u and y. The boundary conditions can be simplified through the use of the correct scaling factors. If the free stream velocity function $U(x)$ is used as the scaling factor for $u(x,y)$, the dimensionless velocity u_1 varies from zero at the wall, to one across the boundary-layer. The scale factor of y can be designated as $a(x)$, where $a(x)$ is made proportional to the boundary-layer thickness. Stated quantitatively, the requirement for similarity is

$$\frac{u\left[x_1, \dfrac{y}{a(x_1)}\right]}{U(x_1)} = \frac{u\left[x_2, \dfrac{y}{a(x_2)}\right]}{U(x_2)} \tag{2.11}$$

where the subscripts refer to different positions of x.

By introducing two scaling factors a(x) and b(x) such that

$$\eta = \frac{y}{a(x)}, \quad f(\eta) = \frac{\Psi(x,y)}{b(x)} \tag{2.12}$$

the velocity components from equation (2.7) become

$$u = \Psi_y = \frac{b(x)}{a(x)} f_\eta(\eta) \tag{2.13}$$

$$v = -\Psi_x = b(x)\left[\frac{a'(x)}{a(x)}\eta f_\eta(\eta) - \frac{b'(x)}{b(x)} f(\eta)\right] \tag{2.14}$$

where the prime and subscript denote differentiation. If one lets

$$b(x) = a(x)U(x) \tag{2.15}$$

the boundary conditions (2.9) and (2.10) can easily be found as:

$$f(0) = 0, \quad f_\eta(0) = 0 \tag{2.16}$$

$$f_\eta(\eta \rightarrow \infty) \rightarrow 1 \tag{2.17}$$

Equations (2.13) and (2.14) can now be substituted in the boundary-layer equation (2.8) to yield

$$UU'f_\eta^2 - U^2\frac{b'}{b} f f_{\eta\eta} = UU' + \frac{M}{\rho} n \frac{U^{2n+1}}{b^{n+1}} f_{\eta\eta}^{n-1} f_{\eta\eta\eta}$$

This equation can be reduced to the form

$$f_{\eta\eta\eta} + \frac{\rho b^n b'}{MnU^{2n-1}} f f_{\eta\eta} + \frac{\rho U' b^{n+1}}{MnU^{2n}} (1 - f_\eta^2) f_{\eta\eta}^{1-n} = 0 \tag{2.18}$$

Equation (2.18) indicates a similar solution only if the resulting equation is an ordinary differential equation. This condition implies that

$$\frac{\rho b^n b'}{MnU^{2n-1}} = \alpha = \text{constant} \tag{2.19}$$

and

$$\frac{\rho U' b^{n+1}}{MnU^{2n}} = \beta = \text{constant} \tag{2.20}$$

It is evident from equation (2.18) that there are four possible combinations of α and β:

$$\alpha = 0, \quad \beta = 0$$

$$\alpha \neq 0, \quad \beta = 0$$

$$\alpha = 0, \quad \beta \neq 0$$

$$\alpha \neq 0, \quad \beta \neq 0$$

If both α and β are zero, equation (2.18) becomes

$$f_{\eta\eta\eta} = 0$$

Two integrations with respect to η of the above relation yields

$$f_\eta = C\eta + D$$

where C and D are integration constants to be found by applying the boundary conditions. However, by using the boundary condition

(2.17) it can be shown that the constants C and D are undefined. Therefore, no similar solution exists for the case α and β both equal to zero.

The second combination is $\alpha \neq 0$ and $\beta = 0$. If $\beta = 0$, then from relation (2.20) either $U'(x)$ or $b^{n+1}(x)$ must be equal to zero. Now from equation (2.15), $U(x)$ is a function of $b(x)$ and if $b(x)$ were equal to zero then $U(x)$ must also equal zero. This solution is trivial however, and therefore $U'(x)$ must be equal to zero. This condition implies that $U(x)$ must be a constant and equation (2.18) then becomes

$$f_{\eta\eta\eta} + f f_{\eta\eta}^{2-n} = 0 \tag{2.21}$$

This is the equation of boundary-layer flow of power-law fluids past a semi-infinite flat plate and has been solved by Hsu [7] by the same method presented in the next chapter.

The third possible combination is that $\alpha = 0$ and $\beta \neq 0$. Relation (2.19) indicates that the scale factor $b(x)$ is a constant and equation (2.20) becomes

$$U^{-2n} \frac{dU}{dx} = \frac{\beta Mn}{\rho b^{n+1}} = B_1 = \text{constant} \tag{2.22}$$

If $n = 1/2$, then equation (2.22) imples that

$$U(x) = Ce^{B_1 x} \tag{2.23}$$

where C is a constant. This is relation for spiral flows and was solved by Hayasii [6] in closed form. If $n \neq 1/2$, the result of education (2.22) is

$$U(x) = \left[B_1 \quad (1-2n)x + B_2\right]^{\frac{1}{1-2n}}$$

It is desirable, however, to obtain a free stream velocity U(x) so that it can be applied to a variety of fluids. The above equation indicates that the velocity U(x) is a function of the type of fluid. Not much information can be obtained by studying this type of flow and this case will therefore be omitted from the analysis.

The last possible combination of α and β is that both are non-zero constants. The result obtained by integration of relation (2.19) is

$$b(x) = \left[\frac{\alpha Mn(n+1)}{\rho} \int_0^x U^{2n-1} dx + A_1\right]^{1/(n+1)} \tag{2.24}$$

Substituting (2.24) into (2.20) the result obtained is

$$\beta = \frac{\rho U'}{MnU^{2n}} \left[\frac{\alpha Mn(n+1)}{\rho} \int_0^x U^{2n-1} dx\right]^{1/(n+1)} \tag{2.25}$$

Since β must be a constant, the right side of equation (2.25) must be a constant, or independent of x. It can be shown that this condition is satisfied if

$$U(x) = A(B+Cx)^q \tag{2.26}$$

where A, B, C, and q are arbitrary constants. By substitution of equation (2.26) into equation (2.25) the result obtained is

$$\beta = \frac{q(n+1)}{q(2n-1)+1} \tag{2.27}$$

Now, by letting $\alpha = 1$, equation (2.18) reduces to

$$f_{\eta\eta\eta} + ff_{\eta\eta}^{2-n} + \beta(1 - f_{\eta}^{2})f_{\eta\eta}^{1-n} = 0 \tag{2.28}$$

where β is given by equation (2.27). When B = 0 and $q > 0$, equation (2.26) describes the flow past a wedge of angle equal to $2q\pi/(q+1)$. This type of flow is known as Falkner-Skan type flow and will be the type studied here.

The boundary conditions to be satisfied by equation (2.28) are

$$f(0) = 0, \quad f_{\eta}(0) = 0 \tag{2.29}$$

$$f_{\eta}(\eta \rightarrow \infty) \rightarrow 1 \tag{2.30}$$

Using the similarity transformations (2.12) through (2.15) and relation (2.24), the shear stress defined by the power-law model may be written as

$$\tau_{xy} = U^2\left[n(n+1)\right]^{\frac{-n}{n+1}} f_{\eta\eta}^{n}\left[Re\right]^{-1/(n+1)} \tag{2.31}$$

where Re is the dimensionless Reynolds number and can be found to be equal to

$$Re = \frac{U^{2-n}(B/C+x)^{n}\rho}{M(2nq-q+1)^{n}} \tag{2.32}$$

where $U = A(B+Cx)^{q}$.

Acrivos *et al* [1] placed a limitation on the value for n used in the Reynolds number. He noted that if the Reynolds number was large (i.e. boundary-layer flow), then $n \leq 2$. This is evident from equation (2.32). If values for $n > 2$ were allowed, the Reynolds number could become small, and as a result the boundary-layer equations would not be valid. For this reason, this study limits n such that $n \leq 2$.

The coefficient of skin friction C_d is defined as

$$C_d = \frac{Re^{1/(n+1)}}{U^2} \left[\tau_{xy} \right]_{y=0}$$

$$= \left[n(n+1)\right]^{-n/(n+1)} \quad f^n_{\eta\eta}(0) \qquad (2.33)$$

CHAPTER III

METHOD OF SOLUTION

According to Mekseyn [10], a solution of equation (2.28) might be obtained if $f(\eta)$ is first expanded in a series in powers of η such that for small values of η

$$f(\eta) = \sum_{m=0}^{\infty} \frac{a_m \eta^m}{m!} \tag{3.1}$$

By applying the boundary conditions (2.29), substituting the series (3.1) into equation (2.28), and collecting terms of equal power of η, the coefficients of a_m in the series of (3.1) can be found to be

$$\left.\begin{aligned}
a_0 &= 0 \\
a_1 &= 0 \\
a_2 &= a \\
a_3 &= -\beta a^k \\
a_4 &= \beta^2 k a^{2k-1} \\
a_5 &= -\beta^3 k(2k-1) + (2\beta-1) a^{k+2} \\
a_6 &= \beta^4 k(6k^2-7k+2) + \beta\left[2k(2\beta+1)-6\beta+4\right] a^{2k+1} \\
a_7 &= -\beta^5 k(24k^3-46k^2+26k-6) a^{5k-4} \\
&\quad + \beta^2\left[(28\beta-18)k^2+(18\beta-17)k+6\beta-4\right] a^{3k} \\
&\text{etc.}
\end{aligned}\right\} \tag{3.2}$$

Notice that the coefficients in relations (3.2) are expressed in terms of known parameters β and $k = 1-n$, and the unknown coefficient $a_2 = a$.

For sufficiently small values of η it can be found from relations (3.1) and (3.2) that

$$ff^k_{\eta\eta} = \sum_{m=0}^{\infty} \frac{e_m \eta^m}{m!} \tag{3.3}$$

with the coefficients e_m defined as

$$\left.\begin{aligned}
e_0 &= 0\\
e_1 &= 0\\
e_2 &= a^{k+1}\\
e_3 &= -\beta(3k+1)a^{2k}\\
e_4 &= \beta^2 k(12k-1)a^{3k-1}\\
e_5 &= -\beta^3 k(60k^2-43k+9)a^{4k-2} + (2\beta-1)(10k+1)a^{2k+2}\\
e_6 &= \beta^4 k(360k^3-522k^2+267k-48)a^{5k-3}\\
&\quad -\left[\beta^2(60k^2+26k+6) - \beta(90k^2+28k+4)\right] a^{3k+1}\\
e_7 &= -\beta^5 k(2520k^4-5646k^3+4875k^2-1917k+288)a^{6k-4}\\
&\quad + \beta^2\left[\beta(1008k^3+42k^2+284k+6)-1008k^3-46k^2\right.\\
&\qquad \left. -108k-4\right] a^{4k}\\
&\text{etc.}
\end{aligned}\right\} \tag{3.4}$$

These coefficients (3.4) are needed in evaluating the unknown coefficients a_2 by the method of steepest descent.

Equation (2.28) can be written as

$$f_{\eta\eta\eta} + G(\eta)f_{\eta\eta} = H(\eta) \tag{3.5}$$

where $G(\eta)$ and $H(\eta)$ are given by

$$G(\eta) = ff^k_{\eta\eta} \tag{3.6}$$

$$H(\eta) = \beta(f^2_\eta - 1)f^k_{\eta\eta} \tag{3.7}$$

The integrating factor I.F. for equation (3.5) is

$$\text{I.F.} = e^{\int G(\eta)d\eta} \tag{3.8}$$

The result obtained by integrating (3.5) once is

$$f_{\eta\eta}(\eta) = e^{-F(\eta)}\phi(\eta) \tag{3.9}$$

where

$$F(\eta) = \int_0^\eta ff^k_{\eta\eta}\, d\eta \tag{3.10}$$

$$\phi(\eta) = f_{\eta\eta}(0) + \beta\int_0^\eta e^{F(\eta)}(f^2_\eta - 1)f^k_{\eta\eta}\, d\eta \tag{3.11}$$

By integrating (3.9) once more and applying the boundary condition $f_\eta(0) = 0$ the result is

$$f_\eta(\eta) = \int_0^\eta e^{-F(\eta)}\phi(\eta)d\eta \tag{3.12}$$

and this integral appears to be of the form that can be evaluated by the method of steepest descent.

To evaluate integral (3.12), $F(\eta)$ is first expanded in a power series of η such that

$$F(\eta) = \eta^3 \sum_{m=0}^{\infty} c_m\eta^m = \tau \tag{3.13}$$

It is noted here that the first term of (3.13) starts with η^3 and from equations (3.3) and (3.10) the coefficient c_m is found to be

$$c_m = \frac{e_{m+2}}{(m+3)!} \tag{3.14}$$

By use of the inversion theorem [13], it can be shown from equation (3.13) that

$$\eta = \sum_{m=0}^{\infty} \frac{b_m}{m+1} \tau^{(m+1)/3} \tag{3.15}$$

and

$$d\eta = \frac{1}{3} \sum_{m=0}^{\infty} b_m \tau^{(m-2)/3} d\tau \tag{3.16}$$

For a given value of m, equations (3.15) and (3.16) can be used to show that

$$\oint^{0+} \frac{d\eta}{\tau^{(m+1)/3}} = \frac{b_m}{3} \oint^{0+,0+,0+} \frac{d\tau}{\tau} = 2\pi i b_m \tag{3.17}$$

where $\oint^{0+}$ is a single circuit around the zero point $\eta = 0$, and this corresponds to a triple circuit (i.e. $\oint^{0+,0+,0+}$) around $\tau = 0$. Cauchy's integral theorem [4] can be applied to (3.17) to show that b_m is the residue in the expression $\tau^{-(m+1)/3}$, or more specifically, b_m is the coefficient of η^{-1} in the expression $\tau^{(m+1)/3}$. The coefficent b_m can now be found to be the coefficient of η^m in the expression

$$\left[c_0+c_1\eta+c_2\eta^2+c_3\eta^3 \;\ldots\ldots\right]^{-(m+1)/3} = \sum_{j=0}^{\infty} D_j(m)\eta^j \qquad (3.18)$$

or that

$$b_m = D_m(m) \qquad (3.19)$$

The coefficient $D_j(m)$ in relation (3.18) can be found by Maclaurin series expansion. The results are:

$$\left.\begin{aligned}
D_0(m) &= c_0^{-(m+1)/3} = D_0 \\
D_1(m) &= D_0 P_1 r_1 \\
D_2(m) &= D_0\left[P_2\frac{r_1}{2!} + P_1 r_2\right] \\
D_3(m) &= D_0\left[P_3\frac{r_1^3}{3!} + P_2 r_1 r_2 + P_1 r_3\right] \\
&\;\;\vdots \\
D_7(m) &= D_0\left[P_7\frac{r_1^7}{7!} + P_6\frac{r_1^5 r_2}{5!} + P_6\left(\frac{r_1^4 r_3}{4!} + \frac{r_1^3 r_2^2}{3!2!}\right)\right. \\
&\quad +P_4\left(\frac{r_1^3 r_4}{3!} + \frac{r_1^2 r_2 r_3}{2!} + \frac{r_1 r_2^3}{3!}\right) \\
&\quad +P_3\left(\frac{r_1^2 r_5}{2!} + r_1 r_2 r_4 + \frac{r_1 r_3^2}{2!} + \frac{r_2^2 r_3}{2!}\right) \\
&\quad \left. +P_2\left(r_1 r_6 + r_2 r_5 + r_3 r_4\right) + P_1 r_7\right] \\
&\text{etc.}
\end{aligned}\right\} \qquad (3.20)$$

in which $r_j = c_j/c_0$ and P_j is a factorial polynomial of degree j and is defined as

$$P_j = P_j(m) = w(w-1)(w-2).............(w-j+1) \tag{3.21}$$

and

$$w = -(m+1)/3 \tag{3.22}$$

There is no difficulty in recording the expression for $D_j(m)$ for any value of j, since equations (3.20) contain some systematic relations between the subscripts of P, and the power and subscript of r in each term.

Now for sufficiently small values of η , $\phi(\eta)$ can be written as

$$\phi(\eta) = \sum_{j=0}^{\infty} d_j \eta^j \tag{3.23}$$

and d_j can be found from relations (3.1), (3.13), and (3.23) as

$$\left.\begin{aligned}
d_0 &= a_2 = a \\
d_1 &= -\beta a^k \\
d_2 &= \tfrac{1}{2}\beta^2 k a^{2k-1} \\
d_3 &= \tfrac{1}{6}\quad \beta^3 k(2k-1)a^{3k-2} + 2\beta a^{k+2} \\
d_4 &= \tfrac{1}{24}\left\{\beta^4 k(6k^2-7k-2)a^{4k-3} + \beta\left[2\beta(2k-3)-k-1\right]\right\}a^{2k+1} \\
d_5 &= \tfrac{1}{120}\left\{-\beta^5 k(24k^3-46k^2+29k-6)a^{5k-4}\right. \\
&\qquad \left.+\beta^2\left[\beta(28k^2+18k+6)-6k^2+7k+1\right]\right\}a^{3k}
\end{aligned}\right\} \tag{3.24}$$

etc.

By expressing the integral (3.12) in terms of the result is

$$f_\eta(\eta) = \int_0^\eta e^{-F(\eta)}\phi(\eta)d\eta = \int_0^\tau e^{-\tau}\phi(\eta)\frac{d\eta}{d\tau}d\tau \qquad (3.25)$$

where

$$\phi(\eta)\frac{d\eta}{d\tau} = \tau^{-2/3}\sum_{m=0}^{\infty} h_m \tau^{m\,2/3} \qquad (3.26)$$

The expansion (3.26) starts with the term $\tau^{-2/3}$, since η, from relation (3.15), starts with $\tau^{1/3}$.

The expression for h_m may be found through the use of Cauchy's integral theorem. In a similar process as was done in finding the coefficient b_m, h_m can be shown equal to 1/3 the coefficient of η^m in the expression [9]

$$(c_0 + c_1\eta + c_2\eta^2 + \ldots)^{-(m+1)/3}(d_0 + d_1\eta + d_2\eta^2 + \ldots.)$$

or that h_m is the coefficient of η^m in

$$\frac{1}{3}\sum_{j=0}^{\infty} d_j\eta^j \sum_{j=0}^{\infty} D_j(m)\eta^j \qquad (3.27)$$

from which one can obtain

$$h_m = \frac{1}{3}\sum_{j=0}^{\infty} d_j D_{m-j}(m)$$

The coefficients h_m are:

$$h_0 = \frac{1}{3}(6)^{1/3} \; a^{-(2-k)/3}$$

$$h_1 = \beta(6)^{2/3} \quad (3k+1)a^{(k+4)/3} - a^{(k-2)/3}$$

$$h_2 = -\frac{\beta^2 a^{k-2}}{40} \quad 3k^2+17k+11 \tag{3.28}$$

$$h_3 = \frac{(6)^{1/3}}{3240}\Big\{ 1080a^{(2-k)/3}\left[-\beta(160k-12) + 80k + 8\right]$$

$$+18\beta^3 a^{(5k-10)/3}\left[33k^2-114k-35\right]$$

$$+\beta^4 a^{(8k-3)/3}\left[13{,}797k^3+8525k^2+63k+5\right]\Big\}$$

Finally, by substituting (3.26) into (3.25) one finds that

$$f_\eta(\eta) = \int_0^\tau e^{-\tau}\sum_{m=0}^{\infty} h_m \tau^{(m-2)/3} d\tau = \sum_{m=0}^{\infty} h_m \Gamma_\tau\left(\frac{m+1}{3}\right) \tag{3.29}$$

where Γ_τ is the incomplete gamma function. Since η can be determined from equation (3.15) for any value of τ, relation (3.29) can be used to find the velocity profile after a_2 is determined.

Equation (3.15) is valid for only small values of η. The integral in equation (3.29), however, has a col of order two at $\eta = 0$ and relation (3.15) indicates that the Debye path for the integral is the one in the η-direction with $0 \leq \eta < \infty$.

Now if $\phi(\eta)$ is assumed to be a slowly varying function, then the method of steepest descent can be applied to equation (3.29). It is also important to notice the $F(\eta)$ is a monotonically increasing positive function. Equation (3.29) is therefore a good approximation for the integral in the interval $0 \leq \eta < \infty$. By applying boundary condition (2.30), equation (3.29) becomes

$$f_{\eta}(\eta \rightarrow \infty) = \int_0^{\infty} e^{-\tau}\phi(\eta)\frac{d\eta}{d\tau}\,d\tau = \sum_{m=0}^{\infty} h_m \Gamma\left(\frac{m+1}{3}\right) = 1 \qquad (3.30)$$

where Γ is the complete gamma function.

The only undetermined coefficient $a_2 = a$ of the problem can now be found by summing a finite number of terms in the series (3.30). In the calculations, however, the series is usually divergent and Euler's transformation can be used to find the correct sum (appendix B).

CHAPTER IV

RESULTS AND DISCUSSIONS

A computer program was written for the first ten terms in expression (3.30).* This series is a function of a parameter given by (2.27), a material parameter n, and the unknown coefficient $a_2 = a$. A trial and error procedure is adopted to find the correct value of the unknown coefficient, and when a_2 is found, the series will sum to one. Since this series is usually divergent Euler's transformation must be used to find the correct sum.

When the external flow velocity is $U(x) = A(B + Cx)^q$ the flow is Falkner-Skan type. The results for this problem are presented in Tables 1 and 2. The term wedge angle used in the tables is related to q by

$$\text{wedge angle} = 2\pi q/(q+1) \tag{4.1}$$

If the external stream velocity is $U(x) = Cx^q$, equation (4.1) describes the real wedge angle.

The value $f_{\eta\eta}(0) = a_2$ found by use of relation (3.30) is presented in Table 1 for various wedge angles and different values of n. Since $f_{\eta\eta}(0)$ is the slope of the velocity profile at the body, Table 1 and Figure 4.1 show that for increasing wedge angle, the slope of the velocity profile increases at the body. This

*Fortran computer program is available upon request from Department of Mechanical Engineering, South Dakota State University, Brookings, South Dakota.

TABLE 1

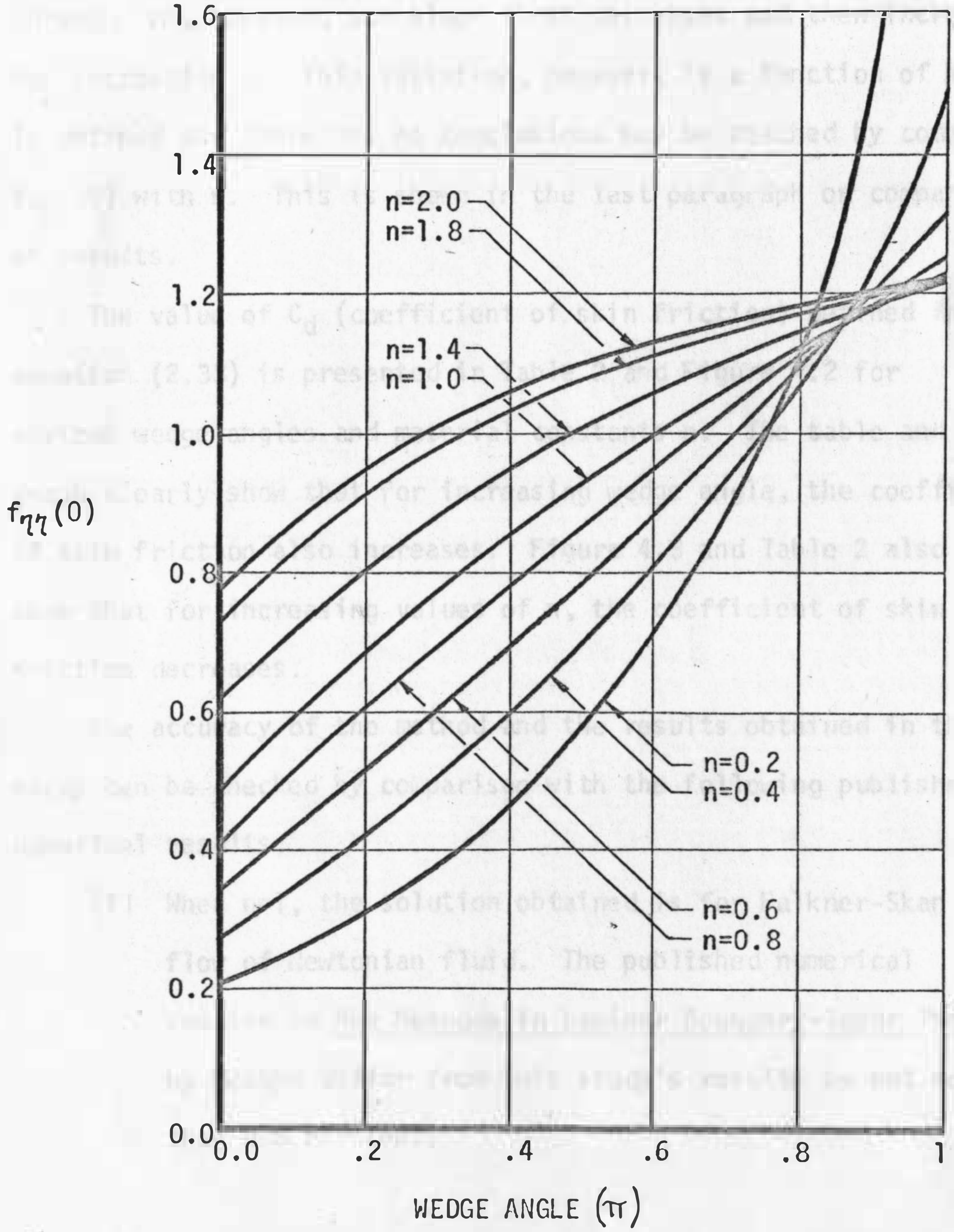
VALUES FOR $f_{\eta\eta}(0)=a$

Wedge Angle	Material Coefficient n						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1π	0.249	0.294	0.339	0.384	0.427	0.470	0.511
0.2π	0.315	0.364	0.415	0.465	0.514	0.561	0.606
0.3π	0.382	0.473	0.497	0.550	0.601	0.650	0.695
0.4π	0.472	0.538	0.605	0.656	0.703	0.746	0.786
0.5π	0.584	0.642	0.698	0.747	0.792	0.831	0.876
0.6π	0.706	0.759	0.806	0.848	0.885	0.918	0.948
0.7π	0.876	0.904	0.933	0.960	0.984	1.006	1.027
0.8π	1.121	1.091	1.083	1.089	1.096	1.098	1.106
0.9π	1.498	1.341	1.266	1.225	1.203	1.191	1.186
π	2.154	1.691	1.492	1.387	1.318	1.289	1.265

Wedge Angle	Material Coefficient n						
	0.9	1.0	1.2	1.4	1.6	1.8	2.0
0.1π	0.548	0.588	0.658	0.713	0.778	0.824	0.873
0.2π	0.647	0.687	0.759	0.812	0.873	0.916	0.940
0.3π	0.738	0.775	0.842	0.874	0.946	0.984	1.010
0.4π	0.819	0.856	0.915	0.936	1.002	1.035	1.062
0.5π	0.899	0.930	0.980	1.017	1.054	1.081	1.104
0.6π	0.974	0.997	1.039	1.072	1.098	1.120	1.138
0.7π	1.043	1.063	1.092	1.115	1.136	1.152	1.166
0.8π	1.116	1.124	1.141	1.156	1.168	1.180	1.187
0.9π	1.185	1.184	1.188	1.193	1.199	1.204	1.209
π	1.250	1.233	1.224	1.225	1.226	1.227	1.228

Figure 4.1

Effect of Parameter q, or Wedge Angle, on $f_{\eta\eta}(0)$ for Various Values of n.



implies that the displacement thickness decreases with increasing wedge angle. Table 1 also shows that for wedge angles of 0.1π through 0.7π the slope of the velocity profile at the body increases with increasing n. From wedge angle values of 0.8π through π, however, the slope first decreases and then increases for increasing n. This variation, however, is a function of how α is defined and therefore no conclusions may be reached by comparing $f_{\eta\eta}(0)$ with n. This is shown in the last paragraph on comparison of results.

The value of C_d (coefficient of skin friction) defined in equation (2.33) is presented in Table 2 and Figure 4.2 for various wedge angles and material constants n. The table and graph clearly show that for increasing wedge angle, the coefficient of skin friction also increases. Figure 4.3 and Table 2 also show that for increasing values of n, the coefficient of skin friction decreases.

The accuracy of the method and the results obtained in this study can be checked by comparison with the following published numerical results:

(1) When n=1, the solution obtained is for Falkner-Skan flow of Newtonian fluid. The published numerical results in <u>New Methods in Laminar Boundary-layer Theory</u> by Meksyn differ from this study's results by not more than 0.5 per cent.

TABLE 2

COEFFICIENT OF SKIN FRICTION C_d

Wedge Angle	Material Coefficient n						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1π	0.961	0.861	0.766	0.682	0.609	0.549	0.497
0.2π	1.007	0.918	0.830	0.751	0.681	0.621	0.570
0.3π	1.046	0.993	0.892	0.816	0.748	0.689	0.636
0.4π	1.092	1.032	0.965	0.891	0.822	0.758	0.701
0.5π	1.139	1.088	1.022	0.951	0.883	0.818	0.759
0.6π	1.183	1.144	1.083	1.014	0.944	0.877	0.815
0.7π	1.235	1.206	1.148	1.078	1.006	0.935	0.869
0.8π	1.298	1.276	1.218	1.149	1.073	0.994	0.922
0.9π	1.375	1.357	1.297	1.218	1.135	1.052	0.975
π	1.479	1.455	1.385	1.296	1.198	1.112	1.026

Wedge Angle	Material Coefficient n						
	0.9	1.0	1.2	1.4	1.6	1.8	2.0
0.1π	0.451	0.416	0.356	0.307	0.278	0.250	0.231
0.2π	0.524	0.486	0.423	0.368	0.335	0.302	0.268
0.3π	0.590	0.548	0.479	0.408	0.381	0.343	0.309
0.4π	0.648	0.605	0.529	0.450	0.417	0.376	0.342
0.5π	0.705	0.658	0.575	0.505	0.452	0.407	0.369
0.6π	0.757	0.705	0.617	0.542	0.483	0.434	0.392
0.7π	0.806	0.752	0.654	0.573	0.510	0.456	0.412
0.8π	0.856	0.795	0.690	0.604	0.533	0.476	0.427
0.9π	0.904	0.837	0.724	0.631	0.556	0.494	0.443
π	0.948	0.872	0.751	0.655	0.576	0.511	0.457

Figure 4.2

Effect of Parameter q (or Wedge Angle) on Coefficient of Skin Friction C_d for Various Values of n.

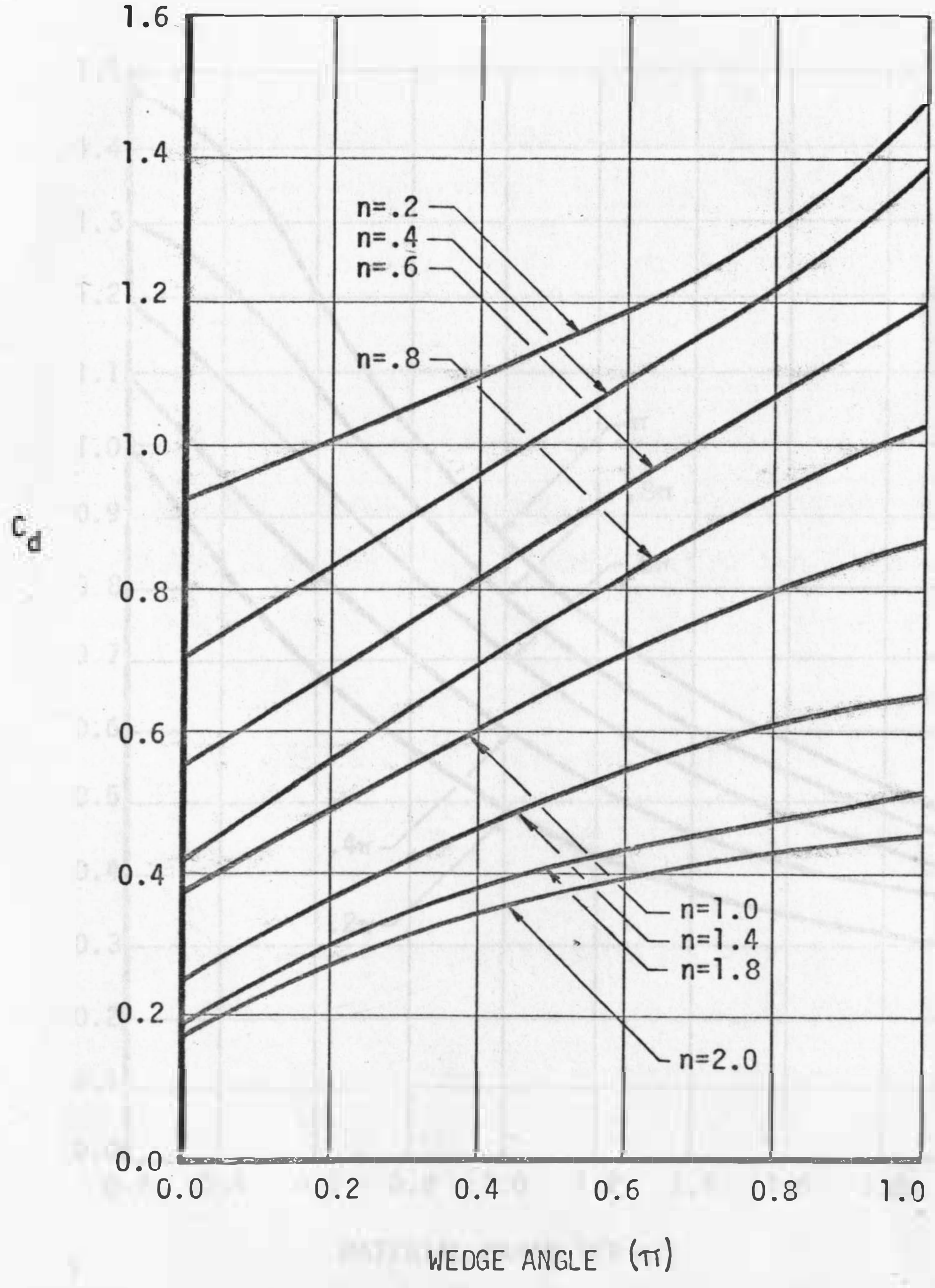
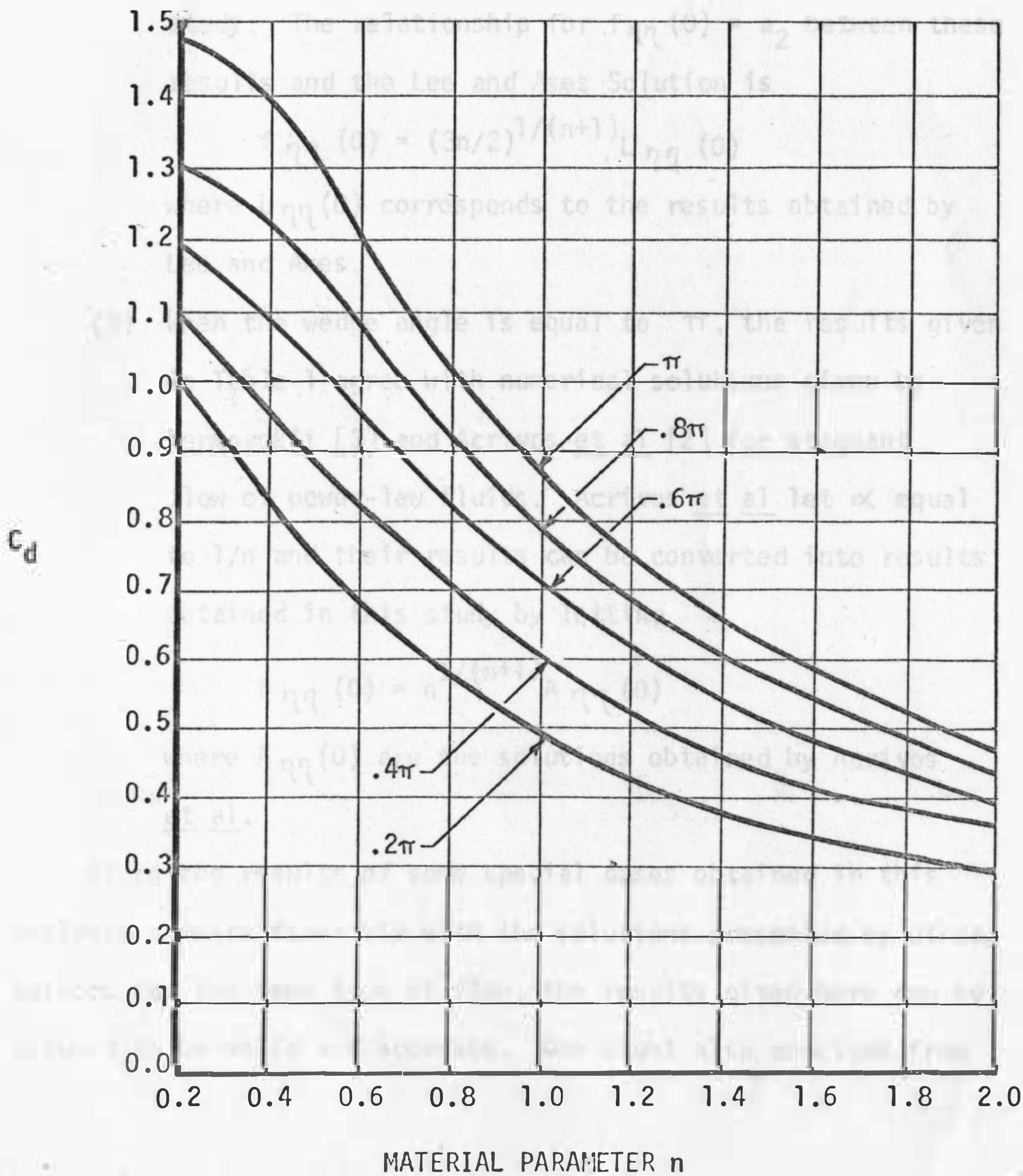


Figure 4.3

Effect of Material Parameter n on Coefficient of Skin Friction C_d for Various Wedge Angles.



(2) The results also agree with numerical solution obtained by Lee and Ames [8]. They studied the flow of power-law fluids past a right angle wedge and their results were presented in a graph. In their analysis they let $\alpha = 2/3n$ instead of $\alpha = 1$ considered in this study. The relationship for $f_{\eta\eta}(0) = a_2$ between these results and the Lee and Ames Solution is

$$f_{\eta\eta}(0) = (3n/2)^{1/(n+1)} L_{\eta\eta}(0)$$

where $L_{\eta\eta}(0)$ corresponds to the results obtained by Lee and Ames.

(3) When the wedge angle is equal to π, the results given in Table 1 agree with numerical solutions given by Berkovskii [3] and Acrivos _et al_ [2] for stagnant flow of power-law fluids. Acrivos _et al_ let α equal to $1/n$ and their results can be converted into results obtained in this study by letting

$$f_{\eta\eta}(0) = n^{1/(n+1)} A_{\eta\eta}(0)$$

where $A_{\eta\eta}(0)$ are the solutions obtained by Acrivos _et al_.

Since the results of some special cases obtained in this analysis compare favorably with the solutions presented by other authors for the same type of flow, the results given here can be assumed to be valid and accurate. One might also conclude from the complexity of this problem and the variety of solutions obtained in this study that the similarity transformation and saddle point method is a powerful technique in solving boundary-layer flow problems.

APPENDIX A

THE METHOD OF STEEPEST DESCENT

The general idea of the saddle point method can be given as follows: Consider the integral

$$F(t) = \int_a^b g(z)e^{th(z)}dz$$

In order to evaluate the integral approximately by the method of steepest descent the integral must meet the following conditions:

(1) $|t|$ is large.

(2) $g(z)$ is a slowly varying function.

(3) The integrand has a col or saddle point of order greater or equal to one.

If the above conditions are met, the points where $h'(z)=0$ are saddle points and the curves at which $\mathrm{Re}\,|th(z)|$ is constant are called level curves. The curves along which $\mathrm{Im}\,|th(z)|$ is constant are called steepest paths. It can be shown from complex variables that the value of the integral is not changed if the integration path is deformed. This is true, however, only if the same endpoints a and b are used.

To evaluate the integral the integration path is deformed such that it coincides as close as possible to the steepest descent path passing through the saddle point. The main contribution to the integral then comes from the region near the col. This method yields a solution in terms of gamma functions.

APPENDIX B

EULER'S TRANSFORMATION

Euler's transformation can be used to find the sum of a divergent series. According to Euler, the sum of a divergent series is the finite numerical value of the convergent expression from which the divergent series was derived [10].

Consider the function

$$\ln(1+x);$$

it has a critical point at $x = -1$, however, the expression

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\ldots,$$

has another critical point at $x = 1$. This critical point is a result of the way the function was expanded and not a result of the function itself. This critical point could have been eliminated had the function been expanded as

$$\ln(1+x) = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1}{3}\left(\frac{x}{1+x}\right)^3 + \ldots\ldots,$$

This procedure is similar to Euler's general case.

Consider the series

$$S(x) = \sum_{m=0}^{\infty} a_m x^{m+1} \tag{B.1}$$

which is convergent for sufficiently small values of x.

Set

$$x = \frac{x}{1-y}, \qquad y = \frac{x}{1+x} \tag{B.2}$$

Substitution of (B.2) into (B.1) and expanding in powers of y results that

$$S(x) = \sum_{n=0}^{\infty} b_n y^{n+1} \qquad (B.3)$$

where

$$b = a_0$$

$$b_1 = a_0 + a_1$$

$$\vdots \qquad \vdots$$

$$b_n = a_0 + \binom{n}{1} a_1 + \binom{n}{2} a_2 + \ldots\ a_n; \qquad (B.4)$$

Equation (B.3) is valid for sufficiently small values of y and as a result of (B.2), it is valid for large values of x. Therefore, while (B.1) diverges for large values of x, (B.3) converges in y and also represents the sum of S(x).

When x = 1, and y = 1/2; (B.3) becomes

$$S(x) = \sum_{n=0}^{\infty} b_n 2^{-(n+1)} \qquad (B.5)$$

and this expression is usually called Euler's transformation.

Euler summed the asymtotic series

$$S(x) = 1!x - 2!x^2 + 3!x^3 - \ldots = \int_0^{\infty} \frac{xte^{-t}}{1+xt}\, dt$$

for x = 1 by applying the transformation (B.5).

The transformation can be applied to yield

$$S(1) = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \ldots$$

Notice, however, that only the first two terms converge.

The sum can be represented by

$$S(1) = \frac{1}{2} - \frac{1}{4} + A$$

where A is the sum of the remaining terms

By applying the transformation to A one finds that

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \dots\dots\dots$$

Divergence is found after the second term in this application. By continually applying this method, the first eight terms sum to S(1) = 0.4008. This is an error of only 0.7 per cent from the expected value of 0.4037. This method could therefore be useful in obtaining the sums of divergent series.

The folowing rules may also be employed when applying Euler's transformation [10].

(1) Include all terms, even zeros, in the transformation.

(2) The transformation may be started from any term, but best results are obtained if the last term in the expansion is the smallest. This will result in better convergence for the same number of transformations.

(3) Repeated transformations not only improve the convergence but also slow it down. It is therefore advisable to use as few transformations as possible.

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